LS-63
T. Khoe
May, 1986

# THE EFFECTS OF ADDING A HIGHER HARMONIC CAVITY

## I. Equation of Motion

The introduction of a higher harmonic cavity permits the control of the synchrotron frequency. In addition, the voltage of the higher harmonic cavity can be chosen such that, for the same bunch length, the frequency spread is larger. Therefore, the Landau damping will be increased, resulting in a higher threshold current.

The phase of the higher harmonic cavity is chosen such that it does not affect the synchronous phase. The equation of motion can then be written in the form

$$\frac{d^2\phi}{df^2} = \Omega_o^2\left[\sin\phi - \sin\phi_s + k\sin m(\phi - \phi_s)\right]$$

where

$$\Omega_o^2 = \left(\frac{c}{R}\right)^2 \left(\frac{h|\eta|eV_1}{2\pi E}\right)$$

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}$$

$V_1$ = peak voltage per turn of the fundamental cavity

$k = \frac{V_m}{V_1}$ and $V_m$ $(m = 2,3)$ is the peak voltage per turn of the $m^{th}$ harmonic cavity

Setting $\psi = \phi - \phi_s$ and expanding $\sin \phi$ and $\sin m(\phi - \phi_s)$ in Taylor's series, we obtain

$$\frac{d^2\psi}{dt^2} = \Omega_o^2\left[\left(\cos \phi_s + mk\right)\psi - 1/2 \sin \phi_s \psi^2 - 1/6\left(\cos \phi_s + m^3k\right)\psi^3 \ldots\right] \qquad (1)$$

Choosing $k = \frac{\alpha^2 - 1}{m} \cos \phi_s$, we find the synchrotron frequency

$$f_s = \frac{\alpha c}{2\pi R} \left[\frac{h|\eta|eV_1|\cos \phi_s|}{2\pi E}\right]^{1/2}$$

Equation (1) can be rewritten in the form

$$\frac{d^2\psi}{dt^2} + \Omega_{so}^2\left[\psi - \varepsilon_1 \psi^2 - \varepsilon_2\psi^3 \ldots\right] = 0 \qquad (2)$$

where $\Omega_{so} = 2\pi f_s$

$$\varepsilon_1 = \frac{1}{2\alpha^2} \tan \phi_s$$

$$\varepsilon_2 = \frac{1}{6} \frac{(m\alpha)^2 - m^2 + 1}{\alpha^2}$$

To calculate the frequency spread as a function of the oscillation amplitude, we introduce in equation (2) the independent variable $\tau$ given by

$$t = \frac{\tau}{\Omega_{so}} \left[1 + h_1\psi_o + h_2\psi_o^2 + \ldots\right],$$

where $\psi_o$ = oscillation amplitude and $h_1$, $h_2$, ... are constants to be determined later.

$$\frac{d^2\psi}{d\tau^2} = -\left(1 + h_1\psi_o + h_2\psi_o^2 + \ldots\right)^2 \left(\psi - \varepsilon_1\psi^2 - \varepsilon_2\psi^3 \ldots\right) \qquad (3)$$

We assume that the periodic solution of this equation can be written in the form

$$\psi = \psi_o \cos \tau + \psi_o^{\,2} x_1(\tau) + \psi_o^3 x_2(\tau)\ldots \tag{4}$$

Here $x_k(\tau)(k = 1,2,\ldots)$ are periodic functions of $\tau$ with period $2\pi$ and the initial conditions $x_k(0) = 0$. Substituting equation (4) in equation (3) and setting the coefficient of each power of $\psi_o$ equal to zero, we find

$$\frac{d^2x_1}{d\tau^2} = -x_1 + \varepsilon_1 \cos^2 \tau - 2h_1 \cos \tau \tag{5A}$$

$$\frac{d^2x_2}{d\tau^2} = -x_2 + 2\varepsilon_1 x_1 \cos \tau + \varepsilon_2 \cos^3 \tau$$

$$+ 2h_1(-x_1 + \varepsilon_1 \cos^2 \tau) - (h_1^{\,2} + 2h_2)\cos \tau. \tag{5B}$$

Equation (5A) and the condition that $x_1(\tau)$ is periodic with $x_1(0) = 0$ give $h_1 = 0$ and $x_1(\tau) = \varepsilon_1(1/2 - 1/3 \cos \tau - 1/6 \cos 2\tau)$. Substituting this in equation (5B), we obtain for a periodic solution of $x_2(\tau)$ with the initial condition $x_2(0) = 0$

$$h_2 = \frac{5}{12}\varepsilon_1^{\,2} + \frac{3}{8}\varepsilon_2$$

and

$$x_2(\tau) = \varepsilon_1^{\,2}\left(-\frac{1}{3} + \frac{29}{144}\cos \tau + \frac{1}{9}\cos 2\tau + \frac{1}{48}\cos 3\tau\right)$$

$$+ \frac{1}{32}\varepsilon_2(\cos \tau - \cos 3\tau)$$

Thus, neglecting terms of higher order than the second in $\psi_o$, we obtain the amplitude dependent frequency

$$\Omega_s = \Omega_{so}[1 - (\frac{5}{12}\varepsilon_1^2 + \frac{3}{8}\varepsilon_2)\psi_o^2 \ldots)$$

or

$$\frac{\Delta f_s}{f_s} = (\frac{5}{12}\varepsilon^2 + \frac{3}{8}\varepsilon_2)\psi_o^2$$

$$= (\frac{5}{48}\frac{\tan^2\phi_s}{\alpha^4} + \frac{m^2}{16} - \frac{m^2-1}{16\alpha^2})\psi_o{}^2$$

The longitudinal single bunch threshold current is proportional to the product of $f_s{}^2$ and $\frac{\Delta f_s}{f_s}$. Table I gives the results for a second harmonic cavity ($m = 2$) and Table II for a third harmonic cavity ($m = 3$). In these tables $(I_{th})_I$ is the threshold current without the higher harmonic cavity and $(I_{th})_{II}$ with the higher harmonic cavity, calculated for the same value of $\psi_o$. The values of $\frac{\Delta E_{rf}}{\sigma_E}$ and the natural bunch length $\sigma_\ell$ are also given in the tables. To find the bucket height, we multiply equation (1) by $\frac{d\phi}{dt}$ and integrate, choosing the integration constant such that $\frac{d\phi}{dt} = 0$ when $\phi = \phi_1$.

$$(\frac{d\phi}{dt})^2 = -2\Omega_o^2[\cos\phi - \cos\phi_1 + (\phi - \phi_1)\sin\phi_s$$

$$+ \frac{k}{m}\{\cos m(\phi - \phi_s) - \cos m(\phi_1 - \phi_s)\}] \qquad (6)$$

$\phi_1$ is an unstable fixed point if it satisfies the condition

$$\sin\phi_1 - \sin\phi_s + k\sin m(\phi_1 - \phi_s) = 0.$$

The trajectory going through $\phi_1$ is a separatrix and the bucket height can be obtained by setting in equation (6) $\phi = \phi_s$.

Table I

(m = 2)

$V_1 = 8.5$ MV, $\phi_s = 135°$, $|\eta| = 3.15 \times \omega^{-4}$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
| k | - | 0.297 | 0.226 | 0.127 | 0 | -0.156 |
| $f_s$(kHz) | - | 1.03 | 1.54 | 2.06 | 2.57 | 3.08 |
| $\frac{(I_{th})_{II}}{(I_{th})_{I}}$ | - | 3.0 | 1.15 | 0.81 | 1 | 1.47 |
| $\frac{\Delta E_{rf}}{\sigma_E}$ | - | 3.1 | 10.7 | 16.1 | 21.1 | 26.0 |
| $\sigma_\ell$(mm) | - | 15 | 10 | 7.5 | 6 | 5 |

Table II

(m = 3)

$V_1 = 8.5$ MV, $\phi_s = 135°$, $|\eta| = 3.15 \times \omega^{-4}$

| $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|---|---|---|---|---|---|---|
| k | 0.226 | 0.198 | 0.151 | 0.089 | 0 | -0.104 |
| $f_s$(kHz) | 0.51 | 1.03 | 1.54 | 2.06 | 2.57 | 3.08 |
| $\frac{(I_{th})_{II}}{(I_{th})_I}$ | 12.8 | 1.45 | 0.05 | 0.14 | 1 | 2.3 |
| $\frac{\Delta E_{rf}}{\sigma_E}$ | 19.4 | 19.5 | 19.8 | 20.3 | 21.1 | 22.5 |
| $\sigma_\ell$(mm) | 30 | 15 | 10 | 7.5 | 6 | 5 |

From this last table, we see that for positive value of k (reducing the synchrotron frequency), the threshold current of the longitudinal single bunch instability can be very small because not only $f_s$ but $\Delta f_s$ is also reduced. Another disadvantage of positive value of k is the inherently long bunch length.